

# Risk-averse model predictive control\*

Domagoj Herceg\*, Pantelis Sopasakis\*\*,†, Alberto Bemporad\* and Panagiotis Patrinos\*\*

**Abstract**—In robust model predictive control (MPC), modeling errors and disturbances are assumed to be unknown-but-bounded quantities and the performance index is minimized with respect to the worst-case realization of the uncertainty (min-max approach). Instead, in stochastic MPC it is assumed that the underlying uncertainty is a random vector following some probability distribution. However, not always can the probability distribution be accurately estimated from available data, nor does it remain constant in time. Using the theory of risk measures, which originated in the field of stochastic finance, we seek to transcend the limitations of robust and stochastic optimal control by proposing a unifying framework that extends and contains both as special cases. We propose risk-averse formulations where the total cost of the MPC problem is expressed as a nested composition of conditional risk mappings. We focus on constrained nonlinear Markovian switching systems and draw parallels between dynamic programming and system theoretic properties to derive Lyapunov-type risk-averse stability conditions. Last, we cast the resulting risk-averse optimal control problem as a second-order cone program which can be solved efficiently.

**Index Terms**—Risk measures; Multistage stochastic programming; Model predictive control; Distributionally robust control.

## I. INTRODUCTION

### A. Motivation

There exist two main ways to deal with uncertainty in model predictive control (MPC), namely, the *robust* and the *stochastic* MPC. In *robust MPC*, modeling uncertainties or disturbances are modeled as unknown-but-bounded quantities and the performance index is minimized with respect to the worst-case realization of the uncertainty (min-max approach) [1], [2]. However, such worst-case events which are unlikely to occur in practice and render robust MPC severely conservative since all statistical information, typically available from past measurements, is completely ignored.

On the other hand, in *stochastic MPC* we assume that the underlying uncertainty is a random vector following some probability distribution [3] and we minimize the expectation of a performance index; such formulations are naturally significantly less conservative. In stochastic MPC, the driving random process is often taken to be normally and independently identically distributed [4] or it is assumed

that it is a finite Markov process [5] and in *scenario-based MPC*, filtered probability distributions are estimated from data [6]–[8]. However, not always can we accurately estimate a probability distribution from available data, nor does it remain constant in time. Stochastic MPC will guarantee mean-square stability of the closed-loop system only with respect to the nominal probability distribution, therefore, errors in the estimation of that distribution may lead to bad performance or even instability.

Using the theory of *risk measures* [9], [10], which sprung from the field of stochastic finance, we seek to transcend the limitations of robust and stochastic optimal control by proposing a unifying framework that extends and contains both as special cases. Roughly speaking, risk measures quantify the importance and effect of the right tail of a distribution of losses, that is, the impact of the occurrence of *extreme events*. The analysis and design of risk-averse MPC controllers was recently identified as a contemporary challenge in stochastic MPC [3]. Risk-averse formulations are of great interest for applications and a number of publications have recently appeared without, however, theoretical stability guarantees [11]–[13].

### B. Background

The first-steps to risk-averse formulations can be traced back to the linear-exponential-quadratic Gaussian control [14] and the study of stochastic control problems under inexact knowledge of the underlying probability distribution which is often termed *distributionally robust* [15], [16]. There have been proposed distributionally robust control methodologies for linear systems with probabilistic constraints assuming knowledge of some moments of the distribution [17], [18]. The same problem was also recently addressed for Markov decision processes with uncertain probabilities [19].

Risk-averse formulations for Markov jump linear systems have been proposed in [20], [21] which amount to the solution of a *linear matrix inequality* (LMI) in real time and leads to high computation times.

### C. Contributions

In this paper we study risk-averse model predictive control formulations for nonlinear Markovian switching systems under — generally nonconvex — joint state-input constraints. We formulate multistage risk-averse optimal control problems using conditional risk measures and draw parallels between dynamic programming and system theoretic properties to derive Lyapunov-type risk-averse stability conditions. When the system is a Markov jump linear system (MJLS)

\* IMT School for Advanced Studies Lucca, Piazza San Francesco 19, 55100 Lucca, Italy.

\*\* KU Leuven, Department of Electrical Engineering (ESAT), STADIUS Center for Dynamical Systems, Signal Processing and Data Analytics, Kasteelpark Arenberg 10, 3001 Leuven, Belgium.

† Corresponding author, pantelis.sopasakis@kuleuven.be.

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with polytopic constraints, we provide a tractable procedure for the design of stabilizing risk-averse controllers.

The solution of multistage risk-averse optimal control problems has been considered prohibitive as only slow cutting-plane methods are currently used [22]–[24]. In Section V we present a computationally tractable approach for the solution of multistage risk-averse problems by casting them as simple second-order cone programs. This formulation renders risk-averse MPC suitable for embedded applications.

Last, we provide simulation examples to showcase the advantages of risk-averse control. Using a cyber-attack scenario, we show that a conventional stochastic MPC design fails to provide mean-square stability if the transition probabilities are inexactly known while the proposed method does stabilize the system in the mean-square sense. We also evaluate risk-averse controllers with different levels of risk aversion on Samuelson's Markovian macroeconomic model.

#### D. Notation

Let  $\bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$  be the set of extended-real numbers,  $\bar{\mathbb{R}}_+$  be the set of positive extended-real numbers,  $\mathbb{N}_{[k_1, k_2]}$  be the integers in  $[k_1, k_2]$ , for  $z \in \mathbb{R}^n$  let  $[z]_+ = \max\{0, z\}$  (where the max is taken element-wise) and for two symmetric matrices  $M_1, M_2$ ,  $M_1 \succcurlyeq M_2$  means that  $M_1 - M_2$  is positive semidefinite. We denote the sets of symmetric positive definite (semidefinite) matrices as  $\mathcal{S}_{++}$  ( $\mathcal{S}_+$ ). We denote the transpose of a matrix  $A$  by  $A^\top$  and the identity matrix by  $I$ . For a set  $C \subseteq \mathbb{R}^n$ , we define its *indicator function* as  $\delta_C(x) = 0$  if  $x \in C$  and  $\delta_C(x) = \infty$  otherwise. The *domain* of an extended-real-valued function  $f : \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$  is  $\text{dom } f = \{x \in \mathbb{R}^n \mid f(x) < \infty\}$ . The *effective domain* of a set-valued mapping  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$  is defined as  $\text{dom } F = \{x \in \mathbb{R}^n \mid F(x) \neq \emptyset\}$ .

## II. RISK-AVERSE OPTIMAL CONTROL

### A. Markovian Switching Systems

In this work we consider Markovian switching systems

$$x_{k+1} = f(x_k, u_k, \theta_k) \quad (1)$$

driven by the random parameter  $\theta_k$  which is a time-homogeneous irreducible and aperiodic Markovian process with values in a finite set  $\mathcal{N} = \{1, \dots, \nu\}$  with transition matrix  $\Pi = (\pi_{ij}) \in \mathbb{R}^{\nu \times \nu}$ , that is  $P[\theta_{k+1} = j \mid \theta_k = i] = \pi_{ij}$ , and *initial distribution*  $v = (v_1, \dots, v_\nu)$  [25]. We denote the *cover* of each mode by  $\mathcal{C}(i) := \{j \in \mathcal{N} \mid \pi_{ij} > 0\}$ . We assume that at time  $k$  we measure the full state  $x_k$  and the value of  $\theta_k$ . As the probabilistic information available up to time  $k$  is fully described by the pair  $(x_k, \theta_k)$ , the control actions  $u_k$  may be decided according to a causal control law of the form  $u_k = \mu_k(x_k, \theta_k)$ .

Function  $f$  is assumed to be continuous in its first two arguments and satisfy  $f(0, 0, i) = 0$ , that is the pair  $(x, u) = (0, 0)$  is an equilibrium point of each subsystem. Markov jump linear systems (MJLS) are a special case of (1) with  $f(x, u, \theta) = A_\theta x + B_\theta u$ .

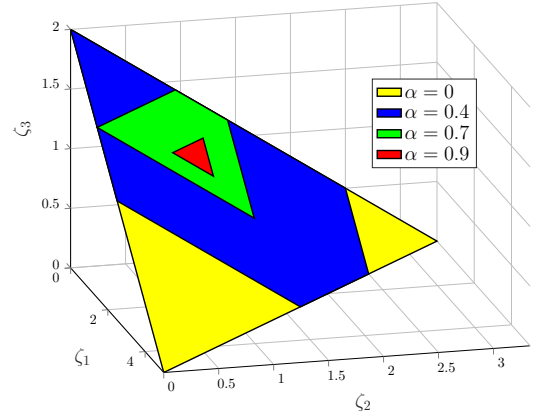


Fig. 1. The admissibility set of  $\text{AV@R}_\alpha$  for different values of  $\alpha$  on a probability space  $\Omega = \{\omega_1, \omega_2, \omega_3\}$  with  $\pi_1 = 0.2$ ,  $\pi_2 = 0.3$  and  $\pi_3 = 0.5$ .

System (1) is subject to the following joint state-input constraints

$$(x_k, u_k) \in Y_{\theta_k}, \quad (2)$$

and we shall assume that for all  $i \in \mathcal{N}$ ,  $Y_i$  are nonempty, closed, convex sets containing the origin in their interiors.

### B. Measuring risk

In what follows  $(\Omega, \mathcal{F}, P)$  is a discrete probability space with  $\Omega = \{\omega_i\}_{i=1}^\kappa$  and let  $\mathcal{Z} = \mathfrak{D}(\Omega, \mathcal{F}, P)$  denote the space of random variables on  $(\Omega, \mathcal{F}, P)$  equipped with the scalar product  $\langle Y, Z \rangle_P = \sum_{i=1}^\kappa \pi_i Y_i Z_i$  where  $Y_i = Y(\omega_i)$ ,  $Z = Z(\omega_i)$  and  $\pi_i = P(\{\omega_i\})$ .

A *risk measure* on  $\mathcal{Z}$  is a mapping  $\rho : \mathcal{Z} \rightarrow \bar{\mathbb{R}}$ . A risk measure  $\rho$  is called *coherent* if it satisfies the following properties [9, Sec. 6.3]

- A1. *Convexity*. For  $Z_1, Z_2 \in \mathcal{Z}$  and  $\lambda \in [0, 1]$ ,  $\rho(\lambda Z_1 + (1 - \lambda)Z_2) \leq \lambda \rho(Z_1) + (1 - \lambda)\rho(Z_2)$ ,
- A2. *Monotonicity*. For  $Z_1, Z_2 \in \mathcal{Z}$  with  $Z_1 \leq Z_2$ ,  $\rho(Z_1) \leq \rho(Z_2)$ ,
- A3. *Translation equivariance*. For  $a \in \mathbb{R}$  and  $Z \in \mathcal{Z}$ ,  $\rho(a + Z) = a + \rho(Z)$ ,
- A4. *Positive homogeneity*. For  $\alpha \geq 0$  and  $Z \in \mathcal{Z}$ ,  $\rho(\alpha Z) = \alpha \rho(Z)$ .

Trivially, the *expectation* operator  $\mathbb{E}(Z) := \sum_{i=1}^\kappa \pi_i Z_i$  is a coherent risk measure and so is the *essential maximum*  $\text{essmax}(Z) := \max\{Z_i; \pi_i > 0\}$ .

A popular risk measure is the *average value-at-risk*, often called *expected shortfall*, which is defined as

$$\text{AV@R}_\alpha(Z) = \begin{cases} \min_{t \in \mathbb{R}} \{t + \alpha^{-1} \mathbb{E}[Z - t]_+\}, & \alpha \in (0, 1], \\ \max\{Z_i; \pi_i > 0\}, & \alpha = 0. \end{cases}$$

$\text{AV@R}_\alpha[Z]$  is the expected value of  $Z$  above its  $1 - \alpha$ -quantile  $Q_{1-\alpha}$ , that is  $\text{AV@R}_\alpha[Z] = \mathbb{E}[Z \mid Z \geq Q_{1-\alpha}]$ .

An important duality result is that all coherent risk measures can be written as

$$\rho(Z) = \max_{\zeta \in C} \langle \zeta, Z \rangle_P = \max_{\zeta \in C} \sum_{i=1}^\kappa \pi_i \zeta_i Z_i, \quad (3)$$

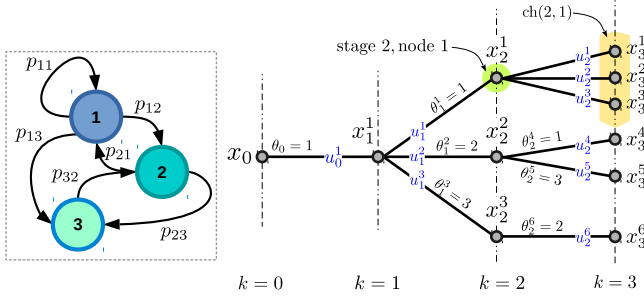


Fig. 2. A Markov chain with three modes and the corresponding tree with  $\theta_0 = 1$ .

where  $C$  is a compact convex set (because  $\rho$  is convex and positively homogeneous) called the *admissibility set* of  $\rho$  whose elements satisfy the properties  $\mathbb{E}[\zeta] = 1$  (because of A3) and  $\zeta_i \geq 0$  (because of A2).

A risk measure is called *polytopic* if its admissibility set is a polytope, i.e., it can be written as the convex hull of a finitely many elements, that is  $C = \text{conv}\{\zeta^{(l)}\}_{l=1}^L$ . Then

$$\rho(Z) = \max_{l \in \mathbb{N}[1, L]} \sum_{j=1}^K \pi_j \zeta_j^{(l)} Z_j.$$

Examples of polytopic risk measures involve the aforementioned average value-at-risk, whose admissibility set is

$$C_\alpha = \{\zeta \in \mathbb{R}^K \mid \mathbb{E}[\zeta] = 1, 0 \leq \zeta_i \leq \frac{1}{\alpha}\}.$$

Note that for  $\alpha = 1$ ,  $C_1 = \{(1, \dots, 1)\}$  and  $\text{AV@R}_1[Z] = \mathbb{E}[Z]$ . The maximal admissibility set is attained for  $\alpha = 0$  and it is  $C_0 = \{\zeta \in \mathbb{R}^K : \mathbb{E}[\zeta] = 1, \zeta_i \geq 0\}$ . As illustrated in Fig. 1,  $\text{AV@R}_\alpha$  interpolates between the risk-neutral expectation operator ( $\text{AV@R}_1 = \mathbb{E}$ ) and the worst-case essential maximum ( $\text{AV@R}_0 = \text{essmax}$ ).

Some other polytopic risk measures are the *mean-upper semideviation*  $\rho[Z] = \mathbb{E}[Z] + c\mathbb{E}[Z - \mathbb{E}[Z]]_+$  with  $c \in [0, 1]$  and, of course, the expectation and the essential maximum.

### C. Scenario Trees and Conditional Risk Mappings

Starting from an initial state  $x_0$  and initial mode  $\theta_0$  and computing control actions according to a causal control law  $u_k = \mu_k(x_k, \theta_k)$ , the future states of the Markovian system, up to some future time  $N$ , span a *scenario tree* — a tree-like structure such as the one shown in Fig. 2.

The pair  $(x_0, \theta_0)$  at stage  $k = 0$  identifies the unique *root node* of the tree. The possible realizations of the system state at time  $k$  define the *nodes* of the tree. The set of all nodes at stage  $k$  defines the set  $\Omega_k$ . Nodes  $a \in \Omega_k$  are identified by pairs  $a = (k, i)$ . The set of nodes in  $\Omega_{k+1}$  which are reachable from a node  $a \in \Omega_k$  is called the set of *children* of  $a = (k, i)$  and is denoted by  $\text{ch}(k, i)$ .

Each node  $a = (k, i) \in \Omega_k$  is uniquely identified by a sequence  $(\theta_0, \dots, \theta_k)$ . As a result,  $P[\{a\}] = \pi_{\theta_0 \theta_1} \pi_{\theta_1 \theta_2} \dots \pi_{\theta_{k-1} \theta_k}$  and this deems  $\Omega_k$  a probability space. We denote the space of random variables  $Z : \Omega_k \rightarrow \mathbb{R}$  as  $\mathcal{Z}_k$ . Spaces  $\mathcal{Z}_k$  can be identified by  $\mathbb{R}^{|\Omega_k|}$ . Likewise, for  $a = (k, i) \in \Omega_k$ , the space  $\text{ch}(a)$  becomes a probability

space where for  $a' = (k+1, j) \in \text{ch}(a)$ ,  $P[\{a'\}] = \pi_{\theta_k \theta_{k+1}}^{i j}$ . Let  $\mathcal{Z}_{k+1}^a$  be the space of random variables on the probability space  $\text{ch}(a)$ ; the risk associated with a  $Z \in \mathcal{Z}_{k+1}^a$  can be measured with a coherent risk measure

$$\rho_k^i : \mathcal{Z}_{k+1}^a \rightarrow \mathbb{R}. \quad (4)$$

Spaces  $\mathcal{Z}_{k+1}^a$  induce a partitioning of  $\mathcal{Z}_{k+1}$ . Indeed,  $\mathcal{Z}_{k+1}$  can be identified by  $\prod_{a \in \Omega_k} \mathcal{Z}_{k+1}^a$ . This partitioning allows us to define mappings  $\rho_k : \mathcal{Z}_{k+1} \rightarrow \mathcal{Z}_k$  of the form

$$\rho_k := (\rho_k^1, \dots, \rho_k^{|\Omega_k|}),$$

where each  $\rho_k^i$  is a coherent risk measure as in (4). Such mappings are called *conditional risk mappings*. An important property of conditional risk mappings which we shall use later is that for all  $Z_{k+1} \in \mathcal{Z}_{k+1}$  and  $Z_k \in \mathcal{Z}_k$

$$\rho_k(Z_k + Z_{k+1}) = Z_k + \rho_k(Z_{k+1}). \quad (5)$$

This is analogous to the translation equivariance property (A3) of risk measures. If  $\rho_k^i$  has the admissibility set  $C_i$ , the collection of sets  $\{C_i\}_i$  is called the *conditional admissibility set* of  $\rho_k$ . We call  $\rho_k$  a *polytopic conditional risk mapping* if all  $C_i$  are polytopes.

### D. Risk-averse Optimal Control and Dynamic Programming

Conditional risk mappings enable us to formulate risk-averse finite-horizon optimal control problems. For a sequence  $(Z_0, \dots, Z_N)$  with  $Z_k \in \mathcal{Z}_k$  and a sequence of conditional risk mappings  $\rho_k : \mathcal{Z}_{k+1} \rightarrow \mathcal{Z}_k$  (assuming that they are all induced by the same risk measure), we define the following mapping  $\varrho : \mathcal{Z}_0 \times \dots \times \mathcal{Z}_N \rightarrow \mathbb{R}$ :

$$\varrho_N(Z_0, \dots, Z_N) = Z_0 + \rho_0(Z_1 + \rho_1(Z_2 + \dots + \rho_{N-1}(Z_N))),$$

which is called a *nested multistage risk measure* [9].

We define the *stage cost* function  $\ell : \mathbb{R}^n \times \mathbb{R}^m \times \mathcal{N} \rightarrow \overline{\mathbb{R}}_+$  and the *terminal cost* function  $\ell_N : \mathbb{R}^n \times \mathcal{N} \rightarrow \overline{\mathbb{R}}_+$ . We hereafter assume that for each  $\theta \in \mathcal{N}$ ,  $\ell(\cdot, \cdot, \theta)$  and  $\ell_N(\cdot, \theta)$  are proper and lower semicontinuous functions with  $\ell(0, 0, \theta) = 0$  and  $\ell_N(0, \theta) = 0$ . We now introduce the following finite-horizon risk-averse optimal control problem

$$\text{minimize}_{u_0, \dots, u_{N-1}} V_N := \varrho_N(\ell(x_0, u_0, \theta_0), \dots, \ell_N(x_N, \theta_N)) \quad (6a)$$

$$\text{subject to } x_0 = x, \theta_0 = i, \quad (6b)$$

$$x_{k+1} = f(x_k, u_k, \theta_k), \text{ for } k \in \mathbb{N}_{[0, N-1]}. \quad (6c)$$

The minimization in (6) is carried out over the causal Markovian control laws  $u_k = \mu_k(x_k, \theta_k)$ . Functions  $\ell$  are extended-real-valued, therefore, they can encode constraints such as (2) by taking  $\text{dom } \ell_k(\cdot, \cdot, \theta) = Y_\theta$ . Likewise,  $\ell_N$  can be used to encode constraints on the terminal state of the form  $x_N \in X_{\theta_N}^f$  by taking  $\text{dom } \ell_N = X^f$ . We hereafter assume that  $X_\theta^f$  and  $Y_\theta$  contain the origin in their interiors.

Problem (6) can be written as a dynamic programming (DP) recursion. For a function  $V : \mathbb{R}^n \times \mathcal{N} \rightarrow \overline{\mathbb{R}}$ , define the dynamic programming operator  $\mathbf{T}$  as [9, Sec. 6.7]

$$\mathbf{T}V(x_k, \theta_k) = \inf_{u_k} \{\ell(x_k, u_k, \theta_k) + \rho_k[V(f(x_k, u_k, \theta_k), \theta_{k+1})]\}$$

where again  $u_k = \mu_k(x_k, \theta_k)$ . Let  $\mathbf{SV}(x_k, \theta_k)$  be the corresponding set of minimizers. This defines the following DP recursion for problem (6)

$$V_{k+1}^* = \mathbf{TV}_k^*, \quad (7a)$$

$$\mathcal{U}_{k+1}^* = \mathbf{SV}_k^*, \quad (7b)$$

for  $k \in \mathbb{N}_{[0, N-1]}$  with  $V_0^*(x, \theta) := \ell_N(x, \theta)$  and  $V_k^* : \mathbb{R}^n \times \mathcal{N} \rightarrow \mathbb{R}^n$  and  $\mathcal{U}_k^* : \mathbb{R}^m \times \mathcal{N} \rightrightarrows \mathbb{R}^m$ . We assume here that all  $\mathcal{U}_k^*(\cdot, i)$  for  $k \in \mathbb{N}_{[1, N]}$  have nonempty effective domain for all  $i \in \mathcal{N}$  — which is the case if  $\ell(x, u, \theta) \geq c\|x\|^2$  for some  $c \geq 0$  and for all  $(x, u) \in Y_\theta$  [26, Prop. 1.17]. Let us stress that it is the use of conditional risk mappings that allows the formulation of a DP recursion.

We may easily verify the monotonicity property

$$\mathbf{TV} \leq \mathbf{TV}', \text{ for all } V, V' \text{ with } V \leq V', \quad (8)$$

following [27]. An observation that will prove useful in what follows is that if  $\mathbf{T}\ell_N \leq \ell_N$ , then  $V_{k+1}^* \leq V_k^*$ .

The above risk-averse optimal control problem leads naturally to the statement of a risk-averse model predictive control problem where control actions are computed by a control law  $\mu_N^*(x, \theta) \in \mathcal{U}_N^*(x, \theta)$ . In Section III we introduce an appropriate risk-based notion of stability and we provide conditions on  $\ell_N$  for the MPC-controlled system  $x_{k+1} = f(x_k, \mu_N^*(x_k, \theta_k), \theta_k)$  to be stable.

### III. RISK-AVERSE STABILITY

Consider the following Markovian switching system which is controlled by  $u_k = \mu(x_k, \theta_k)$

$$x_{k+1} = f_\mu(x_k, \theta_k) := f(x_k, \mu(x_k, \theta_k), \theta_k), \quad (9)$$

subject to the constraints

$$(x_k, \theta_k) \in X^\mu := \{(x, \theta) \mid (x, \mu(x, \theta)) \in Y_\theta\}.$$

**Definition 3.1 (Uniform invariance):** Let  $\mathcal{X} = \{\mathcal{X}_i\}_{i \in \mathcal{N}}$  be a collection of nonempty closed subsets of  $\mathbb{R}^n$  with 0 in their interiors and  $\mathcal{X}_i \subseteq X_i^\mu$ .  $\mathcal{X}$  is called uniformly invariant for (9) under the constraints  $(x_k, \theta_k) \in X^\mu$  if

$$f_\mu(x_k, \theta_k) \in \bigcap_{j \in \mathcal{C}(\theta_k)} \mathcal{X}_j,$$

whenever  $x_k \in \mathcal{X}_{\theta_k}$ .

The concept of uniform invariance is illustrated in Fig. 3. For a collection of sets  $C = \{C_i\}_{i \in \mathcal{N}}$ , let us define the mode-dependent predecessor operator  $R(C) = \{R(C)_i\}_{i \in \mathcal{N}}$  with  $R(C)_i = \{x \in X_i^\mu \mid f(x, i) \in \bigcap_{j \in \mathcal{C}(i)} C_j\}$ . Then  $C$  is a uniformly invariant collection if and only if  $C_i \subseteq R(C)_i$  [5].

Given a uniformly invariant collection of sets  $\mathcal{X}$  and a coherent risk measure  $\rho$ , let  $\rho_k : \mathcal{Z}_{k+1} \rightarrow \mathcal{Z}_k$  be the corresponding conditional risk mappings induced by  $\rho$ , define the coherent risk measure [9, Sec 6.7.3]

$$\bar{\rho}_N = \rho_0 \circ \rho_1 \circ \dots \circ \rho_{N-1}, \quad (10)$$

that is,  $\varrho_N(Z_0, \dots, Z_N) = \bar{\rho}_N(Z_0 + \dots + Z_N)$ . We may now introduce the following stability notion.

**Definition 3.2 (Risk-square stability):** System (9) is *multistage risk-square stable* over a uniformly invariant set

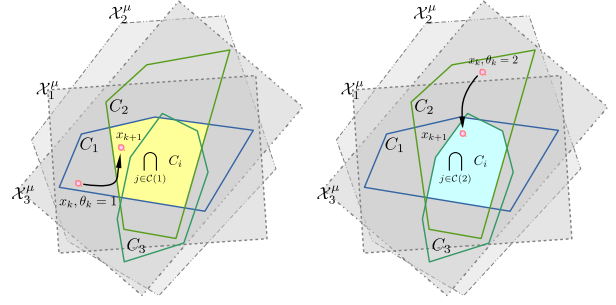


Fig. 3. Illustration of the concept of uniform invariance for a Markovian system with  $\nu = 3$  modes and  $\mathcal{C}(1) = \{1, 2\}$  and  $\mathcal{C}(2) = \{1, 2, 3\}$ . (Left) If  $\theta_k = 1$ ,  $x_{k+1}$  must be contained in  $C_1 \cap C_2$ . (Right) If  $\theta_k = 2$ ,  $x_{k+1}$  must be in  $C_1 \cap C_2 \cap C_3$ .

$\mathcal{X} = \{\mathcal{X}_i\}_{i \in \mathcal{N}}$  if  $\lim_k \bar{\rho}_k(\|x_k\|^2) = 0$ , for all  $x_0 \in \mathcal{X}_{\theta_0}$ . Additionally, it is *multistage risk-square exponentially stable* (mRSES) if  $\bar{\rho}_k(\|x_k\|^2) \leq \lambda \beta^k \|x_0\|^2$ , for some  $\beta \in [0, 1), \lambda \geq 0$ .

The following lemma provides a Lyapunov-type condition for mRSES.

**Lemma 3.3 (mRSES conditions):** Suppose that there exists a  $V : \mathbb{R}^n \times \mathcal{N} \rightarrow \mathbb{R}$  so that for all  $(x_k, \theta_k) \in \text{dom } V$

- 1)  $\text{dom } V$  is a uniformly invariant set
- 2)  $V(x_k, \theta_k) \geq 0$  and  $V(0, \theta_k) = 0$ ,
- 3)  $\alpha_1 \|x_k\|^2 \leq V(x_k, \theta_k) \leq \alpha_2 \|x_k\|^2$ , for some  $\alpha_1, \alpha_2 > 0$ ,
- 4)  $\rho_k(V(x_{k+1}, \theta_{k+1}) - V(x_k, \theta_k)) \leq -c \|x_k\|^2$ , for some  $c > 0$

then, the system is mRSES over  $\text{dom } V$ .

**Proof:** We will use the shorthand notation  $V_k := V(x_k, \theta_k)$ ,  $V_{k+1} := V(x_{k+1}, \theta_{k+1})$ . By virtue of properties 3) and 4) stated above we have

$$\begin{aligned} \rho_k(V_{k+1} - V_k) &\leq -c \|x_k\|^2 \leq -c \alpha_2^{-1} V_k \\ &\leq -\frac{c}{\max\{c, \alpha_2\} + \epsilon} V_k \end{aligned}$$

for any  $\epsilon > 0$ . Because of (5), we have

$$\rho_k(V_{k+1}) \leq \underbrace{\left(1 - \frac{c}{\max\{c, \alpha_2\} + \epsilon}\right)}_{\beta} V_k. \quad (11)$$

Note that  $\beta \in [0, 1)$ . For  $k = 0$  we have  $\rho_0(V_1) \leq \beta V_0$  and  $\rho_1(V_2) \leq \beta V_1$ , so  $\rho_0(\rho_1(V_2)) \leq \beta \rho_0(V_1) \leq \beta^2 V_0$ . Using definition (10) we have  $\bar{\rho}_2(V_2) \leq \beta^2 V_0$  since  $\bar{\rho}_k$  satisfies A4. Recursively we will have

$$\bar{\rho}_k(V_k) \leq \beta^k V_0. \quad (12)$$

Because of condition 2),  $\|x_k\|^2 \leq \frac{1}{\alpha_1} V_k$  and using the positive homogeneity property of  $\bar{\rho}_k$  and (12),

$$\bar{\rho}_k(\|x_k\|^2) \leq \frac{1}{\alpha_1} \bar{\rho}_k(V_k) \leq \frac{1}{\alpha_1} \beta^k V_0 \leq \frac{\alpha_2}{\alpha_1} \beta^k \|x_0\|^2,$$

which completes the proof.  $\blacksquare$

We call a function  $V$  which satisfies the requirements of Lemma 3.3 a (mode-dependent) *risk-averse Lyapunov function*. We may now state conditions on  $\ell$  and  $\ell_N$  which entail mRSES for the risk-averse MPC-controlled system.

**Theorem 3.4 (mRSES):** Suppose the stage cost satisfies  $c\|x\|^2 \leq \ell(x, u, \theta) \leq c'\|x\|^2$  for some  $c' > c > 0$  for

all  $(x, u) \in Y_\theta$  and  $\theta \in \mathcal{N}$  and the terminal cost satisfies  $d\|x\|^2 \leq \ell_N(x, \theta) \leq d'\|x\|^2$ , for some  $d' > d > 0$  for all  $(x, \theta) \in X^f$  and

$$\mathbf{T}\ell_N \leq \ell_N. \quad (13)$$

Then the risk-averse MPC-controlled system  $x_{k+1} = f(x_k, \mu_N^*(x_k, \theta_k), \theta_k)$  is mRSES over  $\text{dom } V_N^*$ .

*Proof:* In light of (8), condition (13) implies  $V_{k+1}^* \leq V_k^*$ , thus  $\text{dom}(V_k^*) \subseteq \text{dom}(V_{k+1}^*) = R(\text{dom } V_k^*)$ , thus  $\text{dom } V_k^*$  is uniformly invariant. By definition  $V_N^*(x_k, \theta_k) = \ell(x_k, \mu_N^*(x_k, \theta_k), \theta_k) + \rho_k[V_{N-1}^*(x_{k+1}, \theta_{k+1})]$ , where  $x_{k+1} = f(x_k, \mu_N^*(x_k, \theta_k), \theta_k)$ . Then

$$\begin{aligned} & \rho_k(V_N^*(x_{k+1}, \theta_{k+1}) - V_N^*(x_k, \theta_k)) \\ &= \rho_k(V_N^*(x_{k+1}, \theta_{k+1})) - V_N^*(x_k, \theta_k) \\ &= \rho_k[V_N^*(x_{k+1}, \theta_{k+1})] - \ell(x_k, \mu_N^*(x_k, \theta_k), \theta_k) \\ &\quad - \rho_k[V_{N-1}^*(x_{k+1}, \theta_{k+1})] \\ &\leq -\ell(x_k, \mu_N^*(x_k, \theta_k), \theta_k) \\ &\leq -c\|x_k\|^2, \end{aligned}$$

where the first equality is because of (5) and the first inequality is because of the monotonicity of  $\rho_k$  and the monotonicity of  $V_k^*$ . Additionally, in light of the quadratic bounds on  $\ell$  and  $\ell_N$ ,  $V_N^*$  satisfies all conditions of Lemma 3.3. ■

In Theorem 3.4 we have shown that if  $\ell$  and  $\ell_N$  are upper- and lower-bounded by quadratic functions and  $\ell_N$  satisfies condition (13), then  $V_N^*$  is a mode-dependent risk-averse Lyapunov function (which need not be quadratic). Note that we do not require that the MPC-controlled system is risk-averse stable with a *quadratic* Lyapunov function.

#### IV. RISK-AVERSE MODEL PREDICTIVE CONTROL FOR MJLS

In this section we provide mRSES conditions and design guidelines for risk-averse MPC of Markov jump linear systems (MJLS) [25], that is  $f(x, u, \theta) = A_\theta x + B_\theta u$ , using a quadratic stage cost  $\ell(x, u, i) = x^\top Q_i x + u^\top R_i u + \delta_{Y_i}(x, u)$ , with  $Q_i \in \mathcal{S}_+$ ,  $R_i \in \mathcal{S}_{++}$  and  $Y_i$  are convex polytopes which contain the origin in their interiors. The terminal cost function is taken to be  $\ell_N(x, i) = x^\top P_i x + \delta_{X_i^f}(x)$  with  $P_i \in \mathcal{S}_{++}$  and  $X_i^f$ . We shall derive conditions on  $P_i$  and  $X_i^f$  so that the stabilizing conditions of Thm. 3.4 are satisfied.

The condition  $\mathbf{T}\ell_N \leq \ell_N$  is equivalent to the following two requirements

$$\begin{aligned} & \min_{u_k} \{ \rho_k[(A_{\theta_k} x_k + B_{\theta_k} u_k)^\top P_{\theta_{k+1}} (A_{\theta_k} x_k + B_{\theta_k} u_k)] \\ & \quad + x_k^\top Q_{\theta_k} x_k + u_k^\top R_{\theta_k} u_k \} \leq x_k^\top P_{\theta_k} x_k, \end{aligned} \quad (14a)$$

$$\text{dom}(\mathbf{T}\ell_N) \supseteq \text{dom } \ell_N \Leftrightarrow R(X^f) \supseteq X^f, \quad (14b)$$

where the minimization in (14a) is over the space of admissible causal control laws  $u_k = \mu_k(x_k, \theta_k)$  so that  $(x_k, \mu_k(x_k, \theta_k)) \in Y_{\theta_k}$ .

Next, we parametrize the controller as  $u(x, i) = K_i x$  and introduce the shorthand notation  $\bar{A}_i = A_i + B_i K_i$  and  $\bar{Q}_i = Q_i + K_i^\top R_i K_i$ . Assuming that  $\rho_k$  is a polytopic conditional risk measure with conditional admissibility set

$C^{(i)} = \text{conv}\{\zeta_i^{(l)}\}_{l \in \mathbb{N}_{[1, \kappa_i]}}$  and using its dual representation, condition (14a) becomes

$$\begin{aligned} & x^\top \bar{Q}_i x + \max_{\zeta_i \in C_i} \sum_{j \in \mathcal{C}(i)} \pi_{ij} \zeta_{ij} x^\top \bar{A}_i^\top P_j \bar{A}_i x \leq x^\top P_i x \\ & \Leftrightarrow \bar{Q}_i + \sum_{j \in \mathcal{C}(i)} \pi_{ij} \zeta_{ij}^{(l)} \bar{A}_i^\top P_j \bar{A}_i \preceq P_i, \end{aligned} \quad (15)$$

for all  $i \in \mathcal{N}$  and  $l \in \mathbb{N}_{[1, \kappa_i]}$ .

Condition (15) can be cast as a linear matrix inequality (LMI) by setting  $P_i^{-1} = Z_i$ ,  $K_i = Y_i Z_i^{-1}$ ,  $F_i^l = [\sqrt{\pi_{i1} \zeta_{i1}^{(l)}} I \dots \sqrt{\pi_{i\nu} \zeta_{i\nu}^{(l)}} I]$  and  $Z = \text{blkdiag}(Z_1, \dots, Z_\nu)$ . The resulting LMI is

$$\begin{bmatrix} Z_i & (A_i Z_i + B_i Y_i)^\top F_i^l & Z_i Q_i^{1/2} & Y_i^\top R_i^{1/2} \\ * & Z_i^l & 0 & 0 \\ * & * & I & 0 \\ * & * & * & I \end{bmatrix} \succcurlyeq 0, \quad (16)$$

for all  $i \in \mathcal{N}$  and  $l \in \mathbb{N}_{[1, \kappa_i]}$ .

Condition (14b) means that  $X^f$  is a uniformly invariant set for the system  $x_{k+1} = (A_{\theta_k} + B_{\theta_k} K_{\theta_k}) x_k$  under the prescribed constraints. Such a set can be determined by the fixed-point iteration  $\mathcal{O}_{k+1} = R(\mathcal{O}_k)$  with  $\mathcal{O}_0 = \{(x, \theta) \mid (x, K_\theta x) \in Y_\theta\}$ . If this iteration converges in a finite number of iterations — a sufficient condition for which is given in [5] — to a set  $\mathcal{O}_\infty$ , this is a *polytopic* uniformly invariant set.

#### V. COMPUTATIONALLY TRACTABLE FORMULATION

Multistage risk-averse problems are considered particularly cumbersome to solve numerically. The main difficulty lies in that  $\bar{\rho}_N$  is the composition of several (typically) nonsmooth functions. Currently only (very slow) cutting plane methods are available which allow the solution of problems with only short prediction horizons and are limited to linear stage cost functions [22]–[24]. An alternative solution approach solves the dynamic programming problem using multiparametric piecewise quadratic programming [28], but its applicability is limited to systems with few states and small prediction horizons [29].

The nested cost function  $V_N$  of the risk-averse optimal control problem introduced in (6a) can be decomposed by defining a sequence of functions  $\Phi_k = \Phi_k(x, u, \theta)$  with

$$\Phi_0 = \rho_{N-1}(\ell_N(x_N, \theta_N))$$

and for  $k \in \mathbb{N}_{[1, N-1]}$

$$\Phi_k = \rho_{N-k-1}(\ell(x_{N-k}, u_{N-k}, \theta_{N-k}) + \Phi_{k-1}).$$

We then have  $V_N = \Phi_{N-1} + \ell(x_0, u_0, \theta_0)$ . Using the definition of AV@R, taking into account the tree structure and employing the epigraphical relaxation trick akin to [30],

$$\begin{aligned} \Phi_0^i &= \min_{t_{N-1}^i} t_{N-1}^i + 1/\alpha \sum_{j \in \text{ch}(N, i)} \pi_{ij} [\ell_N(x_N^i, \theta_N^j) - t_{N-1}^i]_+ \\ &= \min_{\substack{t_{N-1}^i, z_N^{ij} \geq 0 \\ \ell_N(x_N^i, \theta_N^j) - t_{N-1}^i \leq z_N^{ij}}} t_{N-1}^i + 1/\alpha \sum_{j \in \text{ch}(N, i)} \pi_{ij} z_N^{ij}, \end{aligned}$$



$$\underset{\substack{x, u, t \\ z \geq 0}}{\text{minimize}} \ell(x_0^1, u_0^1, \theta_0^1) + \Psi_{N-1}^1 \quad (17a)$$

$$\text{subject to } x_0^1 = x, \theta_0^1 = \theta, \quad (17b)$$

$$\ell_N(x_N^i, \theta_N^j) - t_{N-1}^i \leq z_{N-1}^{ij}, \quad \text{for } i \in \mathbb{N}_{[1, |\Omega_N|]}, j \in \text{ch}(N, i) \quad (17c)$$

$$\ell(x_{N-k}^i, u_{N-k}^j, \theta_{N-k}^j) + \Psi_{k-1}^j - t_{N-k-1}^i \leq z_{N-k}^{ij}, \quad \text{for } k \in \mathbb{N}_{[1, N-1]}, i \in \mathbb{N}_{[1, |\Omega_{N-k}|]}, j \in \text{ch}(N-k, i) \quad (17d)$$

$$\Psi_k^i = t_{N-k-1}^i + 1/\alpha \sum_{j \in \text{ch}(N-k, i)} \pi_{ij} z_{N-k}^{ij}, \quad \text{for } k \in \mathbb{N}_{[1, N-1]}, i \in \mathbb{N}_{[1, |\Omega_{N-k}|]} \quad (17e)$$

$$x_{k+1}^j = A_{\theta_k^j} x_k^i + B_{\theta_k^j} u_k^j, \quad \text{for } k \in \mathbb{N}_{[0, N-1]}, i \in \mathbb{N}_{[1, |\Omega_k|]}, j \in \text{ch}(k, i) \quad (17f)$$

$$(x_k^i, u_k^j) \in Y_{\theta_k^j}, \quad \text{for } k \in \mathbb{N}_{[0, N-1]}, i \in \mathbb{N}_{[1, |\Omega_k|]}, j \in \text{ch}(k, i) \quad (17g)$$

$$x_N^i \in X_{\theta_N^j}^f, \quad \text{for } i \in \mathbb{N}_{[1, |\Omega_N|]}, j \in \text{ch}(N, i) \quad (17h)$$

and similarly

$$\begin{aligned} \Phi_k^i &= \min_{t_{N-k-1}^i} t_{N-k-1}^i + 1/\alpha \sum_{j \in \text{ch}(N-k, i)} \pi_{ij} \left[ \Phi_{k-1}^j - t_{N-k-1}^i \right. \\ &\quad \left. + \ell(x_{N-k}^i, u_{N-k}^j, \theta_{N-k}^j) \right]_+ \\ &= \min_{\substack{t_{N-k-1}^i, z_{N-k}^{ij} \geq 0 \\ \ell(x_{N-k}^i, u_{N-k}^j, \theta_{N-k}^j) + \Phi_{k-1}^j \\ - t_{N-k-1}^i \leq z_{N-k}^{ij}}} t_{N-k-1}^i + 1/\alpha \sum_{j \in \text{ch}(N-k, i)} \pi_{ij} z_{N-k}^{ij}. \end{aligned}$$

This leads to the formulation of the second-order cone program in (17).

This formulation allows us to deconvolve the nested conditional risk functionals and pose the risk-averse optimal control problem as a second-order cone program which can be solved very efficiently online.

## VI. ILLUSTRATIVE EXAMPLE

### A. Resilience to actuator cyber-attacks

The purpose of this example is to demonstrate the effect of inexact knowledge of the probability distribution on the stability properties of the controlled system. Suppose that an attacker tries to alter the normal mode of operation of a system by disconnecting an actuator. We may model this as a Markovian system with two modes:  $i = 1$  corresponds to normal operation and  $i = 2$  corresponds to a successful attack. Suppose we have obtained the following approximate transition matrix from measurements

$$\Pi = \begin{bmatrix} 0.97 & 0.03 \\ 0.03 & 0.97 \end{bmatrix},$$

and the system dynamics is linear described by

$$A_1 = A_2 = \begin{bmatrix} -0.4 & 0.3 \\ 5 & 0.1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

According to the above transition matrix, the attacker has probability 3% to deactivate an actuator and upon a successful attack, the system has 3% probability to recover. Suppose, however, that the attacker has 3.5% probability to gain access to an actuator, that is, the actual — and unknown — probability distribution is described by

$$\Pi' = \begin{bmatrix} 0.965 & 0.035 \\ 0.03 & 0.97 \end{bmatrix}.$$

For  $Q_1 = Q_2 = R_1 = R_2 = I$ , the gain matrices

$$K_1 = \begin{bmatrix} 0.28 & 1.7 \\ -5.2 & 0.29 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0.4 & -0.3 \\ 0 & 0 \end{bmatrix},$$

and the matrices

$$P_1 = \begin{bmatrix} 22.5568 & 0.5563 \\ 0.5563 & 1.957 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 127.7554 & 7.7174 \\ 7.7174 & 4.2507 \end{bmatrix},$$

satisfy the stability Lyapunov condition (16) for  $\alpha = 1$ , therefore, stochastic MPC stabilizes the controlled system in the mean-square sense provided that the probability transition matrix is equal to  $\Pi$ . However, stochastic MPC in practice will fail to lead to a mean square stable closed loop as shown in Fig. 4.

Next, we design a risk-averse model predictive controller with  $\alpha = 0.9$  which encompasses the transition matrix  $\Pi'$ . The risk-averse controller stabilizes the system in the mean-square sense as shown in Fig. 4 (Right). Using the matrix inequality (16) we compute

$$\begin{aligned} K_1 &= \begin{bmatrix} -0.4515 & -0.3112 \\ -3.9806 & -0.0654 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -0.4949 & -0.1910 \\ 0 & 0 \end{bmatrix}, \\ P_1 &= \begin{bmatrix} 25.5573 & 0.4948 \\ 0.4948 & 1.9502 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 184.27 & 11.1038 \\ 11.1038 & 4.6103 \end{bmatrix}. \end{aligned}$$

This example provides clear motivation for risk-averse control as it demonstrates potential vulnerabilities of risk-neutral stochastic MPC formulations and it mitigates the lack of exact probabilistic information.

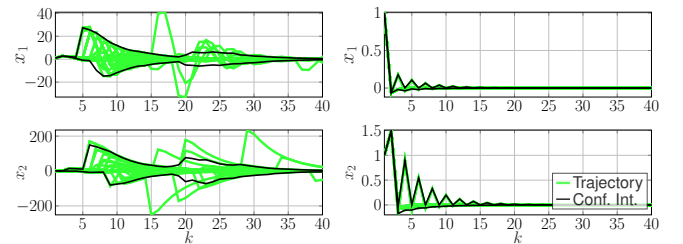


Fig. 4. (Left) Closed-loop simulations with the SMPC controller starting from the initial point  $x_0 = [1, 1]'$  and  $\theta_0 = 1 - 10^4$  random runs. (Right) Closed-loop simulations with the risk-averse MPC controller with  $\alpha = 0.9$ . The black lines denote the bounds of the 99.9% confidence interval.

### B. Samuelson's Economic Model

In this section we apply risk-averse model predictive control to a well-studied MJLS: Samuelson's multiplier-accelerator macroeconomic model [31]. The system has three operating modes with

$$A_1 = \begin{bmatrix} 0 & 1 \\ -2.5 & 3.2 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ -4.3 & 4.5 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 1 \\ 5.3 & -5.2 \end{bmatrix},$$

and  $B_1 = B_2 = B_3 = \begin{bmatrix} 0 & 1 \end{bmatrix}'$  and mode-dependent polyhedral constraints with  $Y_1 = [-10, 10]^3$ ,  $Y_2 = [-8, 8]^2 \times [-10, 10]$ ,  $Y_3 = [-12, 12]^2 \times [-10, 10]$ .

$$Q_1 = \begin{bmatrix} 3.6 & -3.8 \\ -3.8 & 4.87 \end{bmatrix}, Q_2 = \begin{bmatrix} 10 & -3 \\ -3 & 8 \end{bmatrix}, Q_3 = \begin{bmatrix} 5 & -4.5 \\ -4.5 & 5 \end{bmatrix},$$

and  $R_1=2.6$ ,  $R_2=1.165$ ,  $R_3=1.111$ . The estimated transition matrix  $\Pi$  and the actual transition matrix  $\Pi'$  are

$$\Pi = \begin{bmatrix} 0.67 & 0.17 & 0.16 \\ 0.3 & 0.47 & 0.23 \\ 0.26 & 0.1 & 0.64 \end{bmatrix}, \Pi' = \begin{bmatrix} 0.34 & 0.34 & 0.32 \\ 0.3 & 0.3 & 0.4 \\ 0.52 & 0.2 & 0.28 \end{bmatrix}.$$

Risk-averse MPC controllers for different value of  $\alpha$  were designed as discussed in Section IV.

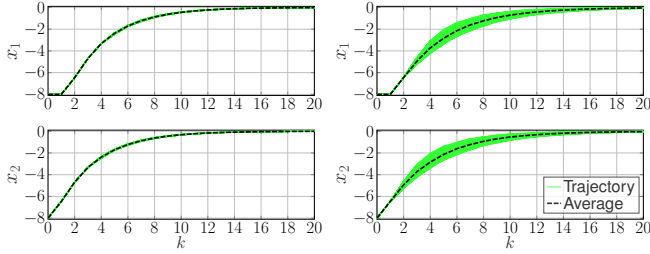


Fig. 5. (Left) Trajectories of the closed-loop system with risk-averse MPC for  $N = 6$  with  $\alpha = 0.1$  and (Right)  $\alpha = 0.5$ . The thin green lines correspond to 1000 random simulations.

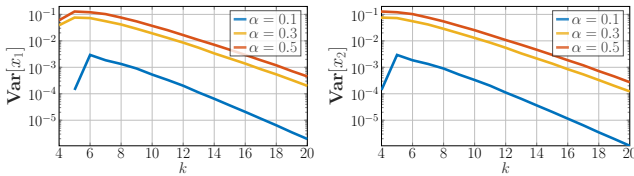


Fig. 6. Estimated variance of  $x_k$  over  $10^4$  runs. (Left) first coordinate of the system state, (Right) second coordinate.

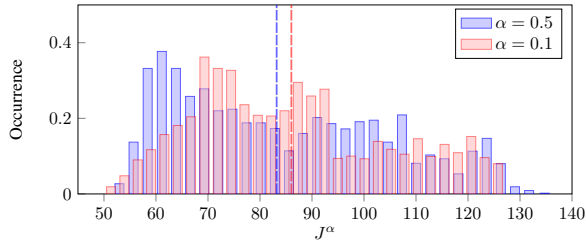


Fig. 7. Histogram of  $J^\alpha$  generated with  $10^4$  runs for  $\alpha = 0.1$  and  $\alpha = 0.5$ . The dashed vertical lines indicate the average of  $J^\alpha$ : For  $\alpha = 0.1$ , the average cost is 86.0, while for  $\alpha = 0.5$  it is 83.2.

As we may observe in Fig. 5, lower values of  $\alpha$  incur a lower risk behavior. This is also show in Fig. 6. In order to further assess the quality of the closed-loop behavior of the

TABLE I  
RUNTIME FOR DIFFERENT PROBLEM SIZES.

$N$	scenarios	mean [s]	max [s].
5	81	0.047	0.052
6	243	0.078	0.084
7	729	0.25	0.29
8	2187	0.89	0.93
9	6561	2.81	2.89

controlled system with different values of  $\alpha$ , we compute  $J^\alpha = \frac{1}{N_s} \sum_{k=0}^{N_s-1} \ell(x_k, u_k, \theta_k)$  over a simulation horizon  $N_s = 50$  for  $10^4$  random runs and we present the histogram of  $J^\alpha$  in Fig. 7. Although lower values of  $\alpha$  lead to a safer operation which can withstand higher risk, it comes at a higher operation cost. Average and maximum computation times for solving the risk-averse optimal control problem are given in Table I for  $\alpha = 0.1$  (similar results are obtained for different values of  $\alpha$ ).

All simulations were performed in MATLAB using YALMIP [32] as a modeling language and the MOSEK solver [33] and were executed on an Intel i5-6200U CPU at 2.30GHz with 8GB RAM running Ubuntu 16.04.

### VII. CONCLUSIONS

The proposed control methodology bridges the gap between min-max and stochastic MPC and furnishes the designers with an additional tuning knob associated with the level of risk the control system should accept. The resulting optimization problem can be formulated as a cone program and solved efficiently enabling its use in embedded applications.

We believe that risk-averse problems possess a favorable structure which can be further exploited to lead to parallelizable implementations akin to ones already developed for stochastic optimal control problems [34]–[36].

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